NATURAL SCIENCES TRIPOS Part IA

Friday 6 June 2008 9 to 12

QUANTITATIVE BIOLOGY

Answer ***four*** *questions from Section A and* ***four*** *questions from Section B.*

You are advised to spend not more than ***one hour*** *on the questions from Section A.*

Attach a ***separate*** *cover sheet to* ***each*** *question.*

STATIONERY REQUIREMENTS
Script Paper
Metric Graph Paper
Rough Work Pads
Blue Coversheets
Tags

SPECIAL REQUIREMENTS
Formulae Booklet
Approved Calculators Allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

A1 A group of organisms living in an isolated habitat is subject to immigration at constant rate α, reproduction at per capita rate β, and death at per capita rate γ, where α, β and γ are positive constants (with $\beta \neq \gamma$).

a) Write down an expression describing the evolution of $Y(t)$, the size of the population at time t.

b) Solve your model to find $Y(t)$, given that the habitat was initially empty.

c) Sketch your solution and biologically interpret the changes to your sketches as the values of the parameters are changed.

d) What happens if $\beta = \gamma$?

A2 The nematode *Ostertagia ostertagi* is one of the most important endoparasites of cattle worldwide, and young calves suffer particularly high parasite loads. The model

$$\frac{dY}{dt} = \alpha e^{-\gamma t} - \beta Y,$$

has been suggested as a suitable model for the (average) parasite load Y suffered by an individual of age t, where α, β and γ are positive parameters (with $\beta \neq \gamma$).

a) Briefly explain the biological basis of the model.

b) Solve the model to find $Y(t)$, given that newborn animals are parasite-free.

c) Calculate the age at which cattle suffer their maximum parasite load, and demonstrate that your expression is dimensionally consistent.

d) Further demonstrate that your expression calculated in part c) predicts a positive age at maximal load for all combinations of the parameters.

A3 An animal gland is placed into a small chamber with volume *V*=5ml, into which pure saline is pumped at a rate of *Q*=1ml per minute. At time *t*=0, the gland is stimulated with cAMP, and thereafter it secretes a hormone at a rate given by $J_m = J_0 e^{-\alpha t}$.

a) Use the integrating factor method to show that the concentration *C*(*t*) of hormone in the efflux from the compartment is given by:

$$C(t) = \frac{J_0}{Q - \alpha V}(e^{-\alpha t} - e^{-\frac{Q}{V}t}).$$

b) If $\alpha = 0.3\,\text{min}^{-1}$ and $J_0 = 5mg\,s^{-1}$, calculate the concentration in the efflux after 1 minute.

A4 Find and classify the stationary points of the function:

$$f(x, y) = (x^2 + y^2)^2 + 2(x^2 - y^2) + 6.$$

A5 A behavioural ecologists wants to know whether peahens prefer peacocks with longer tails. She devises an experiment in which fifteen males have their tail enhanced, reduced, or left unchanged (5 in each treatment). Preference was assessed by counting the number of minutes a female spent in the proximity of each male:

reduced	enhanced	unchanged
12	31	24
11	27	26
12	29	24
13	29	26
14	23	19

a) What are the Null and Alternative Hypotheses?

b) Did the manipulation affect female preference?

c) What assumptions did you have to make for the analysis to be valid?

A6 Two bird species interact according to the pair of differential equations below:

$$\frac{dN_1}{dt} = r_1 N_1 \left[1 - \frac{N_1 + \alpha_{12} N_2}{K_1}\right],$$
$$\frac{dN_2}{dt} = r_2 N_2 \left[1 - \frac{N_2 + \alpha_{21} N_1}{K_2}\right].$$

a) Give a biological interpretation of the model and its parameters.

b) Calculate the null-clines for N_1 and N_2 and plot the null-clines to show only the case where the two species coexist with positive equilibrium populations.

c) Comment briefly on the consequences of increasing the degree of interspecific competition to the persistence of the two populations.

SECTION B

B1 The Giant Hogweed, *Heracleum mantegazzianum*, was foolishly introduced into Great Britain by the ecologically naïve Victorians, and has been spreading extremely rapidly ever since. It causes problems by virtue of its ability to outcompete and therefore displace the native flora, its caustic sap and its immense size. The model

$$\frac{dN}{dt} = \left(b(N) - d(N)\right)N, \qquad (1)$$

has been suggested for $N(t)$, the size of the population of the Hogweed at time t, where $b(N)$ and $d(N)$ are functions of the current population size.

a) Suggest biological interpretations of the functions $b(N)$ and $d(N)$.

b) A particular version of the model has

$$\begin{aligned} b(N) &= b_0 - b_1 N, \\ d(N) &= d_0, \end{aligned} \qquad (2)$$

where $b_0 > d_0$. Explain a plausible biological meaning of these choices of functional form, and in particular explain how the model can be said to exhibit density dependence.

c) Demonstrate that in this case that the model may be written in the standard form of a logistic growth

$$\frac{dN}{dt} = \beta N \left(1 - \frac{N}{\kappa}\right), \qquad (3)$$

if the derived parameters β and κ are chosen in an appropriate fashion. Interpret your expressions for β and κ in the context of your answer to part b).

d) By sketching a suitable graph determine the number of equilibria exhibited by the model, and investigate their stability.

e) Further research suggests that the specification of the model in Equation (2) may be inappropriate, and that instead

$$\begin{aligned} b(N) &= \frac{b_0}{1 + b_1 N}, \\ d(N) &= d_0, \end{aligned}$$

could be a better choice of functional forms in Equation (1). Sketch $b(N)$ and hence interpret the biological basis of the update to the model and suggest one reason why the updated model might be an improvement.

f) Determine the population size at which the total number of Hogweeds is increasing most quickly according to the updated model.

g) Suggest three features of even the updated model that may be unrealistic when modelling the spread of an invading plant species.

B2 A system is represented by the pair of simultaneous differential equations

$$\frac{dx}{dt} = 5 - x^2 - y^2,$$

$$\frac{dy}{dt} = 4x - y^2.$$

a) Find the equilibrium points, and classify them.

b) Sketch the null-clines, and then the phase plane for this system, showing the behaviour around the equilibrium points.

c) For the initial conditions (0,3) sketch the trajectory of the path on the phase plane, and sketch the graph of *x* against *t*, showing all the relevant features.

B3 a) Find the first-order Taylor approximation to

$$f(x,y) = x^3 y - y^2 + x^2 - 2y + 5.$$

b) Find the first-order Taylor approximation to

$$g(x,y,z) = x^3 y - yz + xz - 2y + 5z.$$

c) A solution to the differential equation

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - \frac{dy}{dx} - 3y = 0,$$

is

$$y = Ae^x.$$

Find the complete general solution to the differential equation

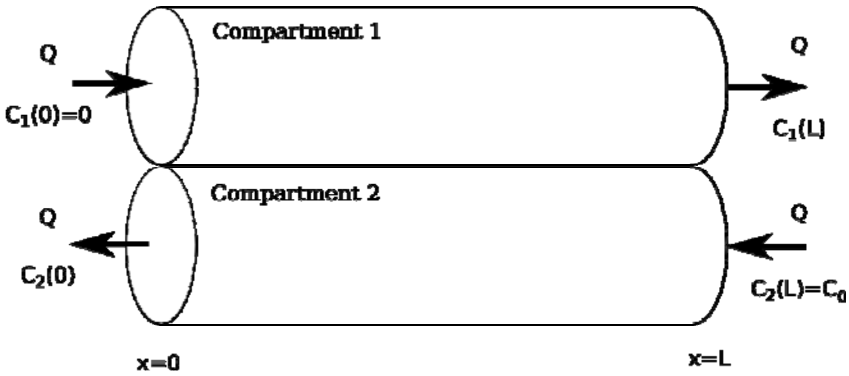
d) Find the particular solution to the differential equation

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - \frac{dy}{dx} - 3y = 9x,$$

with initial conditions

$$\frac{d^2y}{dx^2}(0) = 11, \qquad \frac{dy}{dx}(0) = -8, \qquad y(0) = 4.$$

B4 The following diagram represents a counter current exchanger where material is passed through each compartment in opposite directions at constant volume flow rate Q and transported via a diffusion process from compartment 2 to compartment 1.



a) Given that the secondary flux per unit length between the two compartments $j_m = \gamma[C_2(x) - C_1(x)]$, where γ is a constant, and C_1 and C_2 are the concentrations in the compartments, show that in the steady state the concentration in the first compartment is given by:

$$\frac{dC_1(x)}{dx} = \frac{\gamma}{Q}[C_2(x) - C_1(x)].$$

b) Derive a similar expression for the concentration in the second compartment.

c) Using the identity $\frac{dy}{dx} - \frac{dz}{dx} = \frac{d}{dx}(y - z)$, show that the concentration difference at spatial position x between compartments $C_2(x) - C_1(x)$ is equal to a constant B.

d) Using this result, and the boundary conditions in the diagram above, derive expressions for the concentration profiles in both compartments.

e) Derive an expression for B in terms of Q, L, C_0 and γ.

f) Calculate the total rate R at which material is transferred from compartment 2 to 1.

B5 An ecologist is interested in the relationship between the size of bird nest ferns (kg fresh weight) and the number of insects found within each fern. She collects the following data:

								$\sum x_i$	$\sum x_i^2$
Size of fern	1.2	3.5	5.3	2.8	1.9	2.8	3.1	20.6	70.68
No of insects	50	125	146	107	92	57	101	678	72804

$\sum x_i y_i = 2218.4$

She wants to test whether fern size affects the number of insects found within each fern:

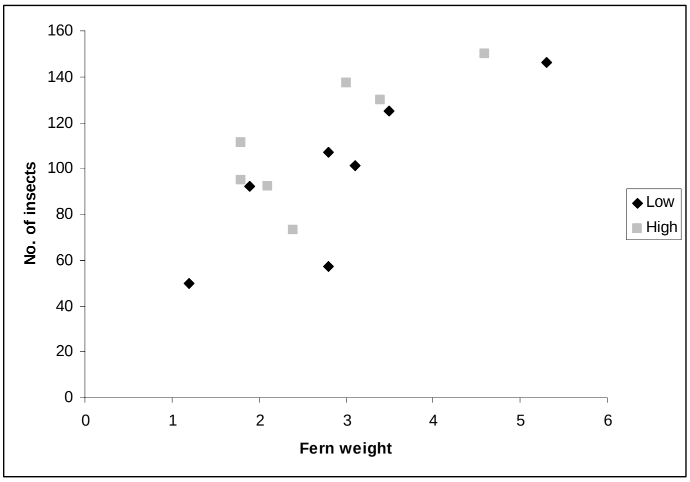
a) What are the Null and Alternative Hypotheses?

b) Find the best linear fit to the relationship between fern size and the number of insects (show your working)

c) Is this relationship significant?

d) What assumptions did you have to make for the analysis to be valid?

The data above were collected for ferns in the low canopy. The ecologist also collected seven ferns from the high canopy and wants to know whether the relationship between fern size and the size of their insect fauna depends on canopy height. She first plots the data:



(TURN OVER)

e) From the model `NoInsects~ Size + Height + Size:Height`, she gets the following coefficients:

```
Coefficients:
(Intercept)          54.946
Size                 21.119
Height (Low)        -23.384
Size:Height (Low)     1.068
```

What is the predicted value for a fern in the low canopy with a weight of 2.3 kg? And what is the predicted value of fern in the high canopy with a weight of 1.9 kg?

f) Dropping the interaction, she gets:

```
Analysis of Variance Table

Model 1: Number ~ Size + Height
Model 2: Number ~ Size + Height+ Size:Height
  Res.Df    RSS Df Sum of Sq
1     11 4000.4
2     10 3996.0  1       4.4
```

Compute F and test whether the interaction is significant. Should she drop the interaction? If yes, what would be the next step in the analysis? If no, what is your conclusion?

B6 In a study of a lizard population, a biologist records the frequencies of three male colour morphs in each of two successive years:

	Male colour morph		
	Orange	**Blue**	**Yellow**
Year 1	24	68	8
Year 2	14	74	12

a) Have the frequencies of the different colour morphs changed over the course of the study?

b) What is the biological implication of your conclusion?

(TURN OVER)

The biologist also records the snout-vent lengths of males of the yellow morph in each year:

Lengths (in mm) of males caught in:

Year 1	Year 2
54	54
65	60
57	58
63	53
54	62
61	52
52	59
51	56
	59
	57
	52
	61

c) Have the lengths of males of the yellow morph changed over the course of the study?

d) What assumptions have you made in your analysis?

B7 a) Write down the equations for the SIR model of a virus in a human population of constant size, N, without vaccination and without births and deaths.

b) Discuss the biological meaning of the parameters.

c) Write down the equation for R_0 and briefly discuss the biological concept, interpretation, and underlying assumptions.

d) Incorporate vaccination into your equation for R_0 from c) and rearrange the equation to determine the proportion of the population that must be vaccinated to prevent an epidemic, p_c, as a function of R_0.

e) Modify the equations from a) to include births and deaths (still without vaccination) and then calculate and sketch the null-clines for S and I. N should still remain constant and everyone should be born susceptible.

B8 Two monkeys are defending a food resource within their group's territory against an intruder. Each must simultaneously decide whether to fight against the intruder or to run away. An individual who fights expends c units of energy. If both fight, the intruder will be successfully repelled, and the two monkeys then divide the food resource, worth b units of energy in total, between them. If only one monkey fights, the intruder will be repelled with probability p (in which case the two monkeys once again divide the resource equally between them), but with probability $(1-p)$ the intruder will succeed in stealing the food resource, in which case the two monkeys gain nothing. If neither monkey fights, then the intruder will certainly steal the resource.

a) Write down a payoff matrix for this game (with payoffs in terms of net expected energetic gain)

b) Under what conditions is each of the pure strategies in this game evolutionarily stable?

c) Under what conditions does the game yield an evolutionarily stable mixed strategy? Derive an expression for the evolutionarily stable probability of fighting under these conditions.

d) Suppose that one monkey is an adult, and one a juvenile. If both fight, the intruder is sure to be repelled, while if neither fight, the intruder is sure to steal the resource. If only the adult fights, then the intruder will be repelled with probability p_1, while if only the juvenile fights, then the intruder will be repelled with probability p_2 ($< p_1$). Write out a new payoff matrix for the game, and determine the conditions under which the strategy 'fight if adult, but not if juvenile' is evolutionarily stable.

e) Suggest three simplistic assumptions of the model that might be modified to render it more realistic.

END OF PAPER