Integrating Factors

We wish to solve equations in the form

$$\frac{dY}{dx} + a(x)Y = f(x) \tag{1}$$

This cannot be solved by any obvious means (such as seperation of variables).

Equation 1 consists of a derivative of Y plus a function of x times Y; this is the kind of result that you get from apply the product rule. This hints at a way of solving the equation; if we can turn the left hand side into a single derivative, we can solve it by simple integration.

We want a solution of the form

$$\frac{d[\mu(x)Y]}{dx} = \mu(x)\frac{dY}{dx} + \frac{d\mu(x)}{dx}Y \tag{2}$$

For some function $\mu(x)$, the integrating factor. By multiplying equation 1 by $\mu(x)$ we get

$$\mu(x)\frac{dY}{dx} + \mu(x)a(x)Y = \mu(x)f(x) \tag{3}$$

If we want to set equation 3 to be equal to equation 2, we need

$$\frac{d\mu(x)}{dx} = \mu(x)a(x) \tag{4}$$

we solve this by seperation of variables

$$\begin{aligned}\int \frac{1}{\mu(x)}d\mu(x) &= \int a(x)dx \\ ln(\mu(x)) &= \int a(x)dx \\ \mu(x) &= e^{\int a(x)dx}\end{aligned} \tag{5}$$

We can now combine equations 2 and 3 to give

$$\begin{aligned}\frac{d[\mu(x)Y]}{dx} &= \mu(x)f(x) \\ \mu(x)Y &= \int \mu(x)f(x)dx \\ Y &= \frac{1}{\mu(x)}\int \mu(x)f(x)dx\end{aligned} \tag{6}$$

which is the integrating factor equation.