The Product Rule and Integration by Parts

1 Product Rule

We want to show that

$$\frac{d[u(x)v(x)]}{dx} = u(x)\frac{dv(x)}{dx} + v(x)\frac{du(x)}{dx} \tag{1}$$

From the definition of the first derivitive

$$\frac{d[u(x)v(x)]}{dx} = \lim_{\Delta x \to 0}\left[\frac{u(x+\Delta x)v(x+\Delta x) - u(x)v(x)}{\Delta x}\right] \tag{2}$$

We can factorise right hand side to give

$$\lim_{\Delta x \to 0}\left[\frac{\{u(x+\Delta x)\left[v(x+\Delta x) - v(x)\right] + u(x+\Delta x)v(x)\} - \{v(x)\left[u(x+\Delta x) - u(x)\right] + v(x)u(x+\Delta x)\}}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{u(x+\Delta x)\left[v(x+\Delta x) - v(x)\right] - v(x)\left[u(x+\Delta x) - u(x)\right]}{\Delta x}\right] \tag{3}$$

$$\tag{4}$$

We can then expand out the limits (using the rules $\lim_{\Delta x \to 0}[a \times b] = \lim_{\Delta x \to 0}[a] \times \lim_{\Delta x \to 0}[b]$ and $\lim_{\Delta x \to 0}[a+b] = \lim_{\Delta x \to 0}[a] + \lim_{\Delta x \to 0}[b]$) to give

$$\lim_{\Delta x \to 0}\left[u(x+\Delta x)\right]\lim_{\Delta x \to 0}\left[\frac{v(x+\Delta x) - v(x)}{\Delta x}\right] - v(x)\lim_{\Delta x \to 0}\left[\frac{u(x+\Delta x) - u(x)}{\Delta x}\right] \tag{5}$$

$u(x+\Delta x)$ tends to $u(x)$, and the other limits are derivitives, thus

$$\frac{d[u(x)v(x)]}{dx} = u(x)\frac{dv(x)}{dx} + v(x)\frac{du(x)}{dx} \tag{6}$$

2 Integration by Parts

To give the Integration by Parts rule, we merely integrate both sides of equation 6 with respect to x

$$\int \frac{d[u(x)v(x)]}{dx}dx = \int \left[u(x)\frac{dv(x)}{dx} + v(x)\frac{du(x)}{dx}\right]dx \tag{7}$$

thus

$$u(x)v(x) + C = \int u(x)\frac{dv(x)}{dx}dx + \int v(x)\frac{du(x)}{dx}dx \tag{8}$$

The constant of integration can be absorbed into the constant of integration for one of the right hand side integrations, and the equation can be rearranged to give

$$\int u(x)\frac{dv(x)}{dx}dx = u(x)v(x) - \int v(x)\frac{du(x)}{dx}dx \tag{9}$$